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APPROXIMATION OF ZEROS OF FUNCTIONS ARISING IN THE ENGINEERING --ETC(U)
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APPROXIMATION OF ZEROS OF FUNCTIONS ARISING IN THE ENGINEERING SCIENCES

THESIS

GCS/MA/77D-3

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GCS/MA/77D-3

6 APPROXIMATION OF ZEROS OF FUNCTIONS

ARISING IN THE ENGINEERING SCIENCES.

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Preface

My undergraduate studies in mathematics and current studies in computer science are combined in this thesis to develop a technique to approximate the zeros of functions which are widely used in the engineering sciences. Used in conjunction with iterative computer algorithms for finding more precise values of the zeros, the results of this thesis should be of value to the United States Air Force in assuring the convergence of these iterative schemes. Except where the theorems are specifically annotated, the work presented herein is my own.

This thesis was sponsored by the Flight Control Division of the Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Ohio. I would like to thank Major Eric Lindberg, Dr. Dan Repperger of the Aerospace Medical Laboratory, Dr. Robert Craig of the Air Force Materials Laboratory, and Professors Constantine Houpis and John D'Azzo of the AFIT Department of Electrical Engineering for taking the time to review this work.

This thesis would not have been possible without the guidance and interest of Dr. John Jones, Jr., whose unflagging optimism and cheerful help inspired and guided me through the successes and the failures. I would also like to thank my typist, Ms. Frances Jarnagin, for accepting this mass of equations and transforming it into a clear, easily readable format. Finally, I thank my wife and daughter for their support, understanding, and patience during our difficult but rewarding tour at AFIT.

John W. Lukes

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Abstract

This thesis extends the works of Morse, Leighton, and Jones which take a function defined by a second order differential equation and determine an interval on which that function either has a zero or attains a minimum, bounded value. Theorems for locating zeros are proved for functions of a single real variable, functions of two real variables, and real vector-valued functions. Algorithms suitable for computer adaptation are presented in examples which locate zeros of such functions occurring in the engineering sciences as Legendre polynomials, *Laguerre polynomials, Emden's equation, and Duffing's equation*; special emphasis is given to the zeros of Bessel functions $J_n(x)$. The methods developed in this work are useful in optimization theory; they also can be used to obtain good initial approximations of zeros for starting iterative algorithms, such as the Newton-Raphson method, which give more exact zero values.

APPROXIMATION OF ZEROS OF FUNCTIONS
ARISING IN THE ENGINEERING SCIENCES

I. Introduction

The location of zeros of polynomials and other functions has always been a matter of great importance in the engineering sciences because these zeros reveal much information concerning the peculiarities of the problem defined by the function. Bessel functions, defined by the differential equation

$$x^2 y'' + xy' + (x^2 - p^2)y = 0 \qquad (y' = \frac{dy}{dx}) \qquad (1.1)$$

have a wide range of applications, from their first use by Bernoulli in determining small oscillations in a hanging chain to the modern problem of flux distribution in a nuclear reactor (Ref 1:312). The Legendre polynomials, which satisfy the differential equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0 \qquad (1.2)$$

have applications in problems in temperature distribution in steady-state heat flow and in both gravitational and electrostatic potential, as well as in other applications of a purely mathematical nature (Ref 1:195).

Another application of finding zeros of functions is in the field of optimization theory. A basic problem in optimization theory is that of determining the variables $x_1, x_2, \ldots, x_n$ which maximize or minimize a function $f(x_1, x_2, \ldots, x_n)$ (Ref 2:269). To do this, the partial derivatives $\frac{\partial f}{\partial x_i}$ are set equal to zero and the resulting n equations

in n unknowns are solved. In other words, the simultaneous zeros of

$$f_{x_i}(x_1, x_2, \ldots, x_n) = \frac{\partial f}{\partial x_i} \quad (i = 1,2,\ldots,n) \tag{1.3}$$

are to be determined.

There exist many iterative schemes (the Newton-Raphson method and the bisection method, for example) which can be used to find the zeros of a function f(x). However, it is often necessary to start the iteration process with an initial value x_o which is already close to the zero. For example, finding the inverse of the number 2 is equivalent to solving the equation

$$\frac{1}{x} - 2 = 0 \tag{1.4}$$

One iterative Newton-Raphson formula for solving this equation is

$$x_{k+1} = x_k(2 - 2x_k) \tag{1.5}$$

An initial value $x_o = 2$ used in Eq (1.5) produces the divergent series $\{2, -4, -40, -3280, \ldots\}$; initial values of $x_o = 1$ and $x_o = 0$ produce the meaningless series $\{1,0,0,\ldots\}$ and $\{0,0,0,\ldots\}$ respectively. However, if $x_o = \frac{1}{4}$ is used to start the iteration, the resulting series $\{\frac{1}{4}, \frac{3}{8}, \frac{15}{32}, \ldots\}$ does converge to the correct solution, $x = \frac{1}{2}$.

The problem of finding good starting values for iterative schemes which find zeros of a function f(x) is considered in this thesis by determining an interval [a,b] which contains at least one zero of f(x). Then, by choosing an x_o from [a,b], perhaps at the midpoint or an endpoint, the point chosen will lie "close" to a zero in that interval. This should give a higher probability of convergence and a faster rate of convergence than an initial value x_o chosen at random.

II. Zeros of Real-Valued Functions of a Single Real Variable

In this chapter, sufficient conditions are established for the existence of a real zero or a bound of a differentiable function f(x) defined on an interval $[a,b]$. Iterative methods for finding zeros of functions often require a close initial approximation in order for the method to converge. However, these initial approximations are often a matter of guesswork and may not converge as desired. The procedures described in this chapter may be used to provide close initial approximations to zeros of f(x); these approximations are then used to start the iterative process, such as the Newton-Raphson method, which will yield more exact results.

2.1 Zeros of f(x) in the Self-Adjoint Second Order Differential Equation $L[f] = 0$

The real zeros of the function f(x) will be considered on a closed interval $[a,b]$, $a < b$. The function f(x) will be required to be a solution of the differential equation of the form

$$L[f] = \frac{d}{dx}\left[A(x)\frac{df}{dx}\right] + C(x)\, f(x) = 0 \tag{2.1}$$

where A(x) is differentiable, $A(x) > 0$, and C(x) is continuous on the interval under consideration.

Use will be made throughout this chapter of admissible trial functions. A function u(x) is defined to be an admissible trial function on $[a,b]$ if u(x) is continuously differentiable, if $u(a) = u(b) = 0$, and if $u(x) > 0$ for $a < x < b$. For example, on $[0,h]$, $h > 0$, one admissible

trial function (and the one used predominantly in this chapter) is

$$u(x) = x(h-x) = xh-x^2 \tag{2.2}$$

Other admissible trial functions which might be used on $[0,h]$ are

$$u(x) = \sin\frac{\pi x}{h} \tag{2.3}$$

or

$$u(x) = x^p\,(h-x)^q \tag{2.4}$$

where $p \geq 1, q \geq 1$. Corresponding admissible trial functions may be adapted to fit any interval $[a,b]$ or $[h,kh]$, $h > 0$ and $k > 1$, to extend the domain of these trial functions.

Associated with the function $f(x)$, the trial function $u(x)$, and the differential equation $L[f]$ will be a functional $J[u]$, where

$$J[u] = \int_0^h \left\{ A(x)\,[u'(x)]^2 - C(x)\,[u(x)]^2 \right\} dx \tag{2.5}$$

Morse (Ref 3 :325-326) proved the following theorem to show that, whenever $J[u]$ can be made negative by varying the parameter h, then a zero must exist on the interval $[0,h]$.

<u>Theorem 2.1 - Morse (Ref 3 :325-326)</u>. If $f(x)$ is a function which satisfies Eq (2.1), and if $u(x)$ is an admissible trial function on $[0,h]$ such that $J[u] < 0$ for the $J[u]$ defined by Eq (2.5), then $f(x)$ has at least one zero on $[0,h]$.

<u>Proof</u>. The proof by contradiction is begun with the assumption that $f(x) \neq 0$ on $[0,h]$. Since $u(0) = u(h) = 0$ by definition, it can be seen that the following identity holds:

$$\int_0^h \left[\frac{A(x)\, f'(x)\, u^2(x)}{f(x)} \right]' dx = \left. \frac{A(x)\, f'(x)\, u^2(x)}{f(x)} \right]_0^h = 0 \tag{2.6}$$

Also, $A(x) > 0$ and the following inequality holds:

$$\left[u'(x) - \frac{u(x)\, f'(x)}{f(x)}\right]^2 \geq 0 \tag{2.7}$$

Therefore, by Eqs (2.6) and (2.7), the following inequalities also hold:

$$0 \leq \int_0^h \left\{ \left[\frac{A(x)\, f'(x)\, u^2(x)}{f(x)} \right]' + A(x) \left[u'(x) - \frac{u(x)\, f'(x)}{f(x)}\right]^2 \right\} dx$$

or (2.8)

$$0 \leq \int_0^h \left\{ \left[A(x)\, f'(x)\right]' \frac{u^2(x)}{f(x)} + A(x)\, f'(x) \left[\frac{u^2(x)}{f(x)} \right]' \right.$$

$$+ A(x) \left[u'(x)\right]^2 - 2\, A(x)\, u'(x)\, \frac{u(x)\, f'(x)}{f(x)}$$

$$\left. + A(x) \left[\frac{u(x)\, f'(x)}{f(x)} \right]^2 \right\} dx$$

which, by substitution from Eq (2.1), is equivalent to

$$0 \leq \int_0^h \left\{ -C(x)\, u^2(x) + A(x)\, f'(x) \left[\frac{2u(x)\, u'(x)\, f(x) - u^2(x)\, f'(x)}{f^2(x)} \right] \right.$$

$$+ A(x) \left[u'(x)\right]^2 - 2A(x)\, u'(x)\, \frac{u(x)\, f'(x)}{f(x)}$$

$$\left. + A(x)\, \frac{u^2(x) \left[f'(x)\right]^2}{f^2(x)} \right\} dx$$

Cancellation of terms produces the following contradiction:

$$0 \leq \int_0^h \left\{ A(x) \left[u'(x)\right]^2 - C(x)\, u^2(x) \right\} dx = J\left[u\right] < 0$$

This contradiction follows from the assumption that $f(x) \neq 0$ on $[0,h]$. Thus, $f(x)$ has at least one zero on the interval $[0,h]$.

Example 2.1. The function $f(x) = \sin x$ is known to be a solution of the differential equation

$$f''(x) + f(x) = 0 \tag{2.9}$$

A comparison to Eq (2.1) gives $A(x) = C(x) = 1$. Then, with the trial function $u(x) = xh - x^2$, h is varied until $J[u] < 0$:

$$J[u] = \int_0^h \left\{ (h - 2x)^2 - (xh - x^2)^2 \right\} dx$$

$$= \int_0^h \left\{ (h^2 - 4hx + 4x^2) - (h^2x^2 - 2hx^3 + x^4) \right\} dx$$

$$= (h^2x - 2hx^2 + \frac{4}{3}x^3 - \frac{1}{3}h^2x^3 + \frac{1}{2}hx^4 - \frac{1}{5}x^5)\Big]_0^h$$

$$= h^3 - 2h^3 + \frac{4}{3}h^3 - \frac{1}{3}h^5 + \frac{1}{2}h^5 - \frac{1}{5}h^5$$

$$= h^3 \left(\frac{1}{3} - \frac{1}{30}h^2\right) < 0$$

This means that a zero will occur on $[0,h]$ when $(\frac{1}{3} - \frac{1}{30}h^2) < 0$, or $h > \sqrt{10} \approx 3.1623$. The actual zero occurs at $x = \pi \approx 3.1416$.

Example 2.2. The function $f(x) = P_n(x)$ is the Legendre polynomial which is a solution to the differential equation

$$(1-x^2) f''(x) - 2x f'(x) + n(n+1) f(x) = \left[(1-x^2) f'(x)\right]' + n(n+1) f(x) = 0 \tag{2.10}$$

Now $A(x) = 1 - x^2 > 0$ for all $|x| < 1$, and $C(x) = n(n+1)$. The trial function $u(x) = xh - x^2$ and Eq (2.5) are used to determine the value h for which $J[u] < 0$.

$$J[u] = \int_0^h \left[(1 - x^2)(h - 2x)^2 - (n^2 + n)(xh - x^2)^2 \right] dx$$

$$= \int_0^h \left[(h^2 - 4hx + 4x^2 - h^2x^2 + 4hx^3 - 4x^4) - (n^2 + n)(h^2x^2 + 2hx^3 - x^4) \right] dx$$

$$= \left(h^2x - 2hx^2 + \frac{4}{3}x^3 - \frac{n^2+n+1}{3}h^2x^3 + \frac{n^2+n+2}{2}hx^4 - \frac{n^2+n+4}{5}x^5\right) \Big]_0^h$$

$$= \frac{h^3}{3} - \frac{h^5}{30}\left[10(n^2 + n + 1) - 15(n^2 + n + 2) + 6(n^2 + n + 4)\right]$$

$$= \frac{h^3}{30}\left[10 - h^2\ (n^2 + n + 4)\right] < 0$$

Therefore, a zero will occur on $[0,h]$ when $h > \left(\frac{10}{n^2 + n + 4}\right)^{\frac{1}{2}}$ and $h < 1$. Table I compares the smallest positive zero of $P_n(x)$ to the value h for several integer values $n \geq 3$, with the true zero being approximated from the generalized Rodrigues' formula (Ref 4:852)

$$P_n(x) = \frac{1}{2^n\ n!}\ \frac{d^{(n)}}{dx^{(n)}}\ (x^2 - 1)^n \qquad (2.11)$$

Table I
Approximation of Zeros of Legendre Polynomials $P_n(x)$

n	Actual Zeros	Approximate Zeros
3	0.7746	0.7906
4	0.3400	0.6455
5	0.5385	0.5423
9	0.3243	0.3262

It is often of interest to find zeros of $f(x)$ on intervals not containing the origin. More generally, the interval $[h,kh]$, $h > 0$ and $k > 1$, may be considered. By changing the limits of integration and adjusting the trial function so that $u(h) = u(kh) = 0$, the following corollary may be established in the same manner as Theorem 2.1.

<u>Corollary</u>. If $f(x)$ is a function which satisfies Eq (2.1), and if $u(x)$ is an admissible trial function on $[h,kh]$ such that $J[u] < 0$ for

$$J[u] = \int_h^{kh} \left\{ A(x) \left[u'(x)\right]^2 - C(x)\, u^2(x) \right\} dx$$

then $f(x)$ has at least one zero on $[h,kh]$.

In this case, a convenient trial function is

$$u(x) = (x - h)(kh - x) = (k + 1)\, hx - x^2 - kh^2 \qquad (2.12)$$

Other trial functions may also be constructed.

<u>Example 2.3</u>. The function $f(x) = J_n(x)$ is the Bessel function of order n of the first kind, which is a solution to the differential equation

$$x^2 f''(x) + x f'(x) + (x^2 - n^2) f(x) = 0 \qquad (2.13)$$

For $x > 0$, Eq (2.13) may be rewritten to match the form of Eq (2.1):

$$x f''(x) + f'(x) + \left(x - \frac{n^2}{x}\right) f(x) = \left[x f'(x)\right]' + \left(x - \frac{n^2}{x}\right) f(x) = 0$$

$$(2.14)$$

Thus, $A(x) = x$ and $C(x) = x - \frac{n^2}{x}$; using $u(x)$ from Eq (2.12), $J[u]$ is calculated:

$$J[u] = \int_h^{kh} \left\{ x\left[(k+1)h - 2x\right]^2 - \left(x - \frac{n^2}{x}\right)\left[(k+1)hx - x^2 - kh^2\right]^2 \right\} dx \tag{2.15}$$

$$= \int_h^{kh} \left\{ x\left[4x^2 - 4(k+1)hx + (k+1)^2h^2\right] - \left(x - \frac{n^2}{x}\right)\left[x^4 - 2(k+1)hx^3 + (k^2+4k+1)h^2x^2 - 2(k^2+k)h^3x + k^2h^4\right] \right\} dx$$

$$= \left(x^4 - \frac{4}{3}(k+1)hx^3 + \frac{1}{2}(k^2+2k+1)h^2x^2 - \frac{1}{6}x^6 + \frac{2}{5}(k+1)hx^5 - \frac{1}{4}(k^2+4k+1)h^2x^4 + \frac{2}{3}(k^2+k)h^3x^3 - \frac{1}{2}k^2h^4x^2 + n^2\left[\frac{1}{4}x^4 - \frac{2}{3}(k+1)hx^3 + \frac{1}{2}(k^2+4k+1)h^2x^2 - 2(k^2+k)h^3x + k^2h^4 \ln x\right]\right)\Bigg]_h^{kh}$$

$$= \Big(h^4(k^4-1) - \frac{4}{3}(k+1)h^4(k^3-1) + \frac{1}{2}(k^2+2k+1)h^4(k^2-1) - \frac{1}{6}h^6(k^6-1) + \frac{2}{5}(k+1)h^6(k^5-1) - \frac{1}{4}(k^2+4k+1)h^6(k^4-1) + \frac{2}{3}(k^2+k)h^6(k^3-1) - \frac{1}{2}k^2h^6(k^2-1)$$

$$+ n^2\left[\frac{1}{4}h^4(k^4-1) - \frac{2}{3}(k+1)h^4(k^3-1) + \frac{1}{2}(k^2+4k+1)h^4(k^2-1) - 2(k^2+k)h^4(k-1) + k^2h^4 \ln k\right]\Big) < 0$$

Combining and simplifying the above expression for J [u] gives

$$h > \left(5 \left\{ \frac{12n^2\, k^2 \ln k + (k^2-1)\left[(2+n^2)(k^2+1) - (4+8n^2)k\right]}{(k-1)^4(k^2-1)} \right\} \right)^{\frac{1}{2}} \qquad (2.16)$$

This inequality cannot easily be solved for k if h is given first. Choosing k first, however, makes it relatively easy to solve Eq (2.16) for h. Thus, to determine a specific h_o for an interval $[h_o, k_o h_o]$, several k values must be chosen experimentally until the k_o is found which produces h_o in Eq (2.16).

It is interesting to examine the size S(h,k) of the interval [h,kh] in the instances where k decreases to 1 (and h becomes very large) and where k becomes very large (and h decreases to 0). Eq (2.16) may be rewritten to express S(h,k), the size of [h,kh]:

$$S(h,k) = kh-h = (k-1)h > \left(5 \left\{ \frac{12n^2k^2\ln k + (k^2-1)\left[(2+n^2)(k^2+1)-(4+8n^2)k\right.}{(k-1)^2(k^2-1)} \right\} \right)^{\frac{1}{2}} \qquad (2.17)$$

As k approaches 1, both the numerator and denominator of the right side of Eq (2.17) approach zero. Applying L'Hospital's Rule three times to the expression within the braces in Eq (2.17) produces the following limit for S(h,k):

$$\lim_{\substack{k\to 1 \\ (h\to\infty)}} S(h,k) = \lim_{\substack{k\to 1 \\ (h\to\infty)}} (kh-h) = \sqrt{10} \qquad (2.18)$$

Thus, for any n, as h gets very large (and k decreases toward 1), the size of the interval containing a zero of $f(x) = J_n(x)$ will become

close to $\sqrt{10}$. This is expressed more exactly by the following corollary:

Corollary. For any $n \geq 0$ and $\epsilon > 0$ there exists a $p > 0$ such that, for all real values $x > p$, there is at least one zero of $J_n(x)$ on the interval $[x, (x + \sqrt{10}) + \epsilon]$.

These results closely correspond to the fact that the distance between zeros of $J_n(x)$ approaches π from below for $n < \frac{1}{2}$ and approaches π from above for $n > \frac{1}{2}$ as x grows very large (Ref 5:49).

Next, by allowing k in Eq (2.17) to grow infinitely large, the limit of the right-hand side again becomes fixed for a given n:

$$\lim_{\substack{k \to \infty \\ (h \to 0)}} S(h,k) = \lim_{\substack{k \to \infty \\ (h \to 0)}} (kh-h) = \left[5\ (2 + n^2)\right]^{\frac{1}{2}} \tag{2.19}$$

This means that, as k grows very large and h approaches 0, the smallest positive zero of $J_n(x)$ lies in the interval $[0, (10 + 5n^2) + \epsilon]$ for an arbitrarily small ϵ. Tables II and III list several intervals containing zeros of $J_n(x)$ obtained by varying the value k in Eq (2.16) and computing the corresponding h. For comparison, the first 10 zeros of $J_n(x)$ for several values of n are assembled in the same tables (Refs 6-10).

McCann proves that $J_{n,p}(x)$, the p^{th} zero of $J_n(x)$, has the lower bound

$$J_{n,p}(x) > \left[(p - \tfrac{1}{4})^2\ \pi^2 + n^2\right]^{\frac{1}{2}} \qquad \text{(Ref 11:102)} \tag{2.20}$$

An algorithm for calculating an upper bound for $J_{n,p}(x)$ can be derived using Eqs (2.19) and (2.16):

(1) Calculate an upper bound $k_1 h_1$ of $J_{n,1}(x)$ using either Eq (2.19) or a k_1 which provides an even smaller bound;

Table II
Comparison of Intervals Containing Zeros to Actual Zeros of Bessel Functions $J_n(x)$, n=0,1,2,3

Interval: $[h,kh]$ = h-kh

k	n = 0	n = 1	n = 2	n = 3
∞	0 - 3.163	0 - 3.873	0 - 5.478	0 - 7.417
50	.064 - 3.227	.077 - 3.882	.107 - 5.390	.144 - 7.234
20	.166 - 3.329	.195 - 3.916	.264 - 5.300	.351 - 7.025
5	.790 - 3.953	.862 - 4.315	1.050 - 5.252	1.304 - 6.521
3	1.581 - 4.744	1.661 - 4.984	1.881 - 5.644	2.199 - 6.599
2	3.162 - 6.325	3.232 - 6.466	3.436 - 6.873	3.750 - 7.501
1.5	6.324 - 9.487	6.375 - 9.563	6.524 - 9.788	6.767 - 10.151
1.1	31.622 - 34.785	31.637 - 34.801	31.680 - 34.849	31.751 - 34.927

Actual Zeros $J_{n,p}(x)$

n	p=1	p=2	p=3	p=4	p=5	p=6	p=7	p=8	p=9	p=10
0	2.405	5.520	8.654	11.792	14.931	18.071	21.212	24.352	27.49	30.63
1	3.832	7.016	10.173	13.324	16.471	19.616	22.760	25.90	29.05	32.19
2	5.136	8.417	11.620	14.796	17.960	21.117	24.270	27.42	30.57	33.71
3	6.380	9.761	13.015	16.223	19.409	22.583	25.75	28.91	32.06	35.22

Table III
Comparison of Intervals Containing Zeros to Actual Zeros of Bessel Functions $J_n(x)$, $n=4,5,10,50$

Interval: $[h,kh]$ = h-kh

k	n = 4	n = 5	n = 10	n = 50
∞	0 - 9.487	0 - 11.619	0 - 22.584	0 - 111.849
50	.184 - 9.216	.225 - 11.263	.436 - 21.822	2.159 - 107.955
20	.444 - 8.895	.541 - 10.834	1.044 - 20.887	5.157 - 103.152
5	1.592 - 7.964	1.900 - 9.504	3.546 - 17.731	17.301 - 86.510
3	2.579 - 7.740	2.999 - 8.998	5.336 - 16.010	25.532 - 76.599
2	4.150 - 8.302	4.614 - 9.230	7.428 - 14.857	33.755 - 67.512
1.5	7.092 - 10.639	7.489 - 11.235	10.216 - 15.325	40.611 - 60.917
1.1	31.851 - 35.037	31.979 - 35.177	33.026 - 36.329	57.169 - 62.887

Actual Zeros $J_{n,p}(x)$

n	p=1	p=2	p=3	p=4	p=5	p=6	p=7	p=8	p=9	p=10
4	7.588	11.065	14.373	17.616	20.827	24.019	27.20	30.37	33.54	36.70
5	8.771	12.339	15.700	18.980	22.218	25.43	28.63	31.81	34.99	38.16
10	14.476	18.433	22.047	25.51	28.89	32.21	35.50	38.76	42.00	45.23
50	57.12	62.81	67.70	72.19	76.44	80.51	84.46	88.32	92.09	95.80

(2) Choose another smaller k_2 and calculate the corresponding value h_2 using Eq (2.16). Adjust this k_2 and recalculate h_2 until h_2 is greater than but close to the upper bound k_1h_1 of the previous interval. Thus, the new interval $[h_2, k_2h_2]$ will contain at least one zero of $J_n(x)$ and will be disjoint from the previous interval.

(3) Repeat step (2) until p disjoint intervals have been created. Since each interval contains at least one zero of $J_n(x)$, the upper limit of the p^{th} interval $[h_p, k_ph_p]$ will be an upper bound of $J_{n,p}(x)$.

Example 2.4. The above algorithm will be used to calculate an upper bound for $J_{3,5}(x)$, the fifth zero of $J_3(x)$. Eq (2.19) may be used to determine that at least one positive zero exists in the interval $I_1 = [h_1, k_1h_1] = [0, (10 + 5n^2)^{\frac{1}{2}}] = [0, 7.417]$. However, using $k_1 = 4.1$ in Eq (2.16) produces the interval $I_1 = [1.583, 6.494]$ which has a much smaller upper bound for $J_{3,1}(x)$. Next, by successively experimenting with Eq (2.16) to get k_i such that $h_i > k_{i-1}h_{i-1}$, the following values of k_i with the associated intervals $[h_i, k_ih_i]$ are obtained:

$$
\begin{aligned}
k_2 &= 1.5234 &&: \quad I_2 = [6.499, 9.900] \\
k_3 &= 1.330 &&: \quad I_3 = [9.923, 13.199] \\
k_4 &= 1.244 &&: \quad I_4 = [13.233, 16.463] \\
k_5 &= 1.194 &&: \quad I_5 = [16.528, 19.735]
\end{aligned}
$$

Each of the intervals I_{1-5} contains at least one zero. Therefore, the upper limit of I_5, 19.735, is an upper bound for $J_{3,5}(x)$. McCann's lower bound in Eq (2.20) for $J_{3,5}(x)$ is 15.221. Therefore

$$15.221 < J_{3,5}(x) < 19.735 \tag{2.21}$$

The exact zero from Table II, $J_{3,5}(x) = 19.409$, is much closer to the upper bound than the lower. However, considerable calculation is involved in determining this upper bound. Even greater precision may be achieved for the upper bound by using more precise k_i's to reduce the distance $(h_i - k_{i-1}h_{i-1})$ between the intervals I_i and I_{i-1}. Since the true distance between zeros for $J_3(x)$ approaches $\pi \approx 3.14$ for large x while this algorithm produces a distance of, at best, $\sqrt{10} \approx 3.16$, there will always be an error of at least $\frac{.02}{3.14}$ no matter how precise k_i is chosen.

Second order differential equations which do not match the form of Eq (2.1) may often be converted to that form by using an integrating factor. For the differential equation

$$p(x)\, f''(x) + q(x)\, f'(x) + r(x)\, f(x) = 0, \; p(x) \neq 0 \tag{2.22}$$

multiplying both sides of the equation by the integrating factor

$$I(x) = \frac{e^{\int [q(x)/p(x)]dx}}{p(x)} \tag{2.23}$$

yields the equation

$$\left[e^{\int [q(x)/p(x)]dx}\, f'(x)\right]' + \frac{r(x)\, e^{\int [q(x)/p(x)]dx}}{p(x)}\, f(x) = 0 \tag{2.24}$$

Thus, the integrating factor can be used to convert Eq (2.22) to the self-adjoint form, Eq (2.24), from which expressions for A(x) and C(x) may be obtained and used in the functional $J[u]$ in Eq (2.5). The

expressions $A(x)$ and $C(x)$ may be too complicated to integrate easily, however, and thus be too unwieldy to use effectively in $J[u]$.

2.2 Zeros of More Complicated Second Order Differential Equations

The introduction of two non-linear functions of $w(x)$, $\alpha(w)$ and $f(w)$, into Eq (2.1) produces the differential equation

$$L[w] = \frac{d}{dx}\left[A(x)\,\alpha[w(x)]\,\frac{dw}{dx}\right] + C(x)\,f[w(x)] = 0 \qquad (2.25)$$

whose solutions have been treated in some detail (Refs 12:377; 13:26-28). The addition of another function $B(x)$ to this equation produces the following more general abbreviated form

$$L[w] = \frac{d}{dx}\left[A(x)\,\alpha(w)\,\frac{dw}{dx}\right] + 2\,B(x)\,\frac{dw}{dx} + C(x)\,f(w) = 0 \qquad (2.26)$$

Methods to establish the existence of real zeros of the function $w(x)$ which satisfies Eq (2.26) are similar to but more involved than those used previously in this chapter.

<u>Theorem 2.2</u>. If the following conditions are true:

(1) the functions, $A(x)$, $\alpha(w)$, and $f(w)$, are differentiable with respect to the variables x and w, where $A(x) > 0$ and $w(x)$ satisfies Eq (2.26), and $B(x)$ and $C(x)$ are continuous on the interval $[a,b]$;

(2) for an admissible trial function $u(x)$ on $[a,b]$, there exists an associated differentiable function $G[u]$ such that $G[u(a)] = G[u(b)] = 0$ and $G[u(x)] > 0$ for $a < x < b$;

(3) $f(w) \neq 0$ if $w(x) \neq 0$;

(4) $g^2(u)\,\alpha(w) \leq 4\, G[u] \frac{df}{dw}$ and $g(u)\,\alpha(w) = 2\,(G[u])^{\frac{1}{2}}$, where

$g(u) = \frac{dG}{du}$; and

(5) $J[u,G] < 0$, with

$$J[u,G] = \int_a^b \left\{ A(x)[u'(x)]^2 - 2B(x)(G[u])^{\frac{1}{2}}\, u'(x) + [Q(x)-C(x)]\, G[u] \right\} dx \tag{2.27}$$

where the function $Q(x)$ makes the matrix

$$M(x) = \begin{bmatrix} A(x) & -B(x) \\ -B(x) & Q(x) \end{bmatrix} \tag{2.28}$$

positive semi-definite on $[a,b]$, which means that, for all real vectors $z = (z_1,z_2)$,

$$z\, M(x)\, z^T = A(x)\, z_1^2 - 2B(x)\, z_1 z_2 + Q(x)\, z_2^2 \geq 0 \tag{2.29}$$

then the function $w(x)$ has at least one zero on $[a,b]$.

<u>Proof</u>. For a proof by contradiction, it is assumed that $J[u] < 0$ but that $w(x) \neq 0$ (and thus $f(w) \neq 0$) on some interval $[a,b]$. If the values $z_1 = u'(x) - \frac{g(u)\,\alpha(w)\,w'(x)}{2f(w)}$ and $z_2 = (G[u])^{\frac{1}{2}}$ are substituted in Eq (2.29), and it is observed that $G[u(a)] = G[u(b)] = 0$ implies that $\int_a^b \left\{\frac{d}{dx} G[u(x)]\right\} dx = G[u(x)]\Big]_a^b = 0$, then the following inequality may be produced:

$$0 \leq \int_a^b \Bigg(\left\{\frac{d}{dx}\left[\frac{A(x)\,\alpha(w)\,G[u]\,w'(x)}{f(w)}\right]\right\} + \Bigg\{ A(x)\left[u'(x) - \frac{g(u)\,\alpha(w)w'(x)}{2f(w)}\right]^2$$
$$- 2B(x)\left[u'(x) - \frac{g(u)\,\alpha(w)\,w'(x)}{2f(w)}\right](G[u])^{\frac{1}{2}} + Q(x)\,G[u] \Bigg\} \Bigg) dx \tag{2.30}$$

Using a shorthand form derived by dropping the variables, Eq (2.30) may be reduced as follows:

$$0 \leq \int_a^b \left(\left\{ \frac{d}{dx}\left[\frac{A\alpha G w'}{f} \right] \right\} + \left\{ A\left[u' - \frac{g\alpha w'}{2f}\right] - 2B\left[u' - \frac{g\alpha w'}{2f}\right]G^{\frac{1}{2}} + QG \right\} \right) dx$$

$$= \int_a^b \left\{ \frac{G}{f}\frac{d}{dx}\left[A\alpha w'\right] + A\alpha w' \frac{d}{dx}\left[\frac{G}{f}\right] + A(u')^2 - A u' \frac{g\alpha w'}{f} \right.$$

$$\left. + A\frac{g^2\alpha^2(w')^2}{4f^2} - 2B u' G^{\frac{1}{2}} + B\frac{g\alpha w'}{f} G^{\frac{1}{2}} + QG \right\} dx$$

$$= \int_a^b \left\{ \frac{G}{f}(-2B w' - Cf) + A\alpha w' \left(\frac{f g u' - f' w' G}{f^2} \right) + A(u')^2 - \frac{A\alpha w' g u'}{f} \right.$$

$$\left. + \frac{A g^2\alpha^2(w')^2}{4f^2} - 2B u' G^{\frac{1}{2}} - B\frac{g\alpha w'}{f} G^{\frac{1}{2}} + QG \right\} dx$$

$$= J\left[u,G\right] + \int_a^b \left\{ \frac{G}{f}(-2B w') + \left(A\alpha w' \frac{g u'}{f} - \frac{A\alpha w' g u'}{f}\right) \right.$$

$$\left. + \left(- \frac{A\alpha (w')^2 f' G}{f^2} + \frac{A g^2\alpha^2(w')^2}{4f^2}\right) + B\frac{g\alpha w'}{f} G^{\frac{1}{2}} \right\} dx$$

$$= J\left[u,G\right] + \int_a^b \left\{ \frac{A(w')^2\alpha}{4f^2}(g^2\alpha - 4G f') + \frac{Bw'}{f} G^{\frac{1}{2}} (g\alpha - 2G^{\frac{1}{2}}) \right\} dx < 0 \tag{2.31}$$

This contradiction is based on assumptions (4) and (5) and on $w(x) \neq 0$ on $[a,b]$; if (4) and (5) can be satisfied, the assumption that $w(x) \neq 0$ must be false. Thus, $w(x)$ has at least one zero on $[a,b]$.

The number of restrictions placed on the various functions in this theorem does limit the range of applications. Several types of problems

do exist, however, which fit the form of $L[w]$ in Eq (2.25).

Example 2.5. The Laguerre polynomials defined by

$$L_n(x) = \frac{e^x}{n!} \frac{d^{(n)}}{dx^{(n)}} [x^n e^{-x}] \tag{2.32}$$

satisfy the second order differential equation

$$x L_n''(x) + (1-x) L_n'(x) + n L_n(x) = 0 \tag{2.33}$$

This may be rewritten

$$[x L_n'(x)]' - x L_n'(x) + n L_n(x) = 0 \tag{2.34}$$

to match Eq (2.25), with $A(x) = x$, $B(x) = -\frac{x}{2}$, $C(x) = n$, $w(x) = L_n(x)$, $\alpha(w) = 1$, and $f(w) = w(x) = L_n(x)$. $Q(x) = \frac{x}{4}$ will make the matrix $M(x)$ in Eq (2.28) positive semi-definite. Finally, if $u(x) = (x-a)(b-x)$ and $G[u] = u^2$, then $g(u) = \frac{dG}{du} = 2u$ and conditions (1-4) of Theorem 2.2 are satisfied. The above information is then substituted into Eq (2.27) to satisfy condition (5):

$$J[u,G] = \int_a^b \Big\{ x\,(a+b-2x)^2 - 2\,(-\tfrac{x}{2})(x-a)(b-x)(a+b-2x) + (\tfrac{x}{4} - n)\,[(x-a)(b-x)]^2 \Big\}\, dx < 0 \tag{2.35}$$

To find an interval containing the first positive zero of $L_n(x)$, the value of a in Eq (2.35) is made to approach zero. It can be seen that, as a decreases toward zero, the value of the integrand in Eq (2.35) evaluated at a will also approach zero. Therefore, with a approaching zero and $b = h$, the limiting value of Eq (2.35) becomes

$$J\left[u,G\right] = \int_0^h \left\{ x(h-2x)^2 + x(xh-x^2)(h-2x) + (\frac{x}{4} - n)(xh-x^2)^2 \right\} dx \qquad (2.36)$$

$$= \int_0^h \left\{ (h^2x - 4hx^2 + 4x^3) + (h^2x^2 - 3hx^3 + 2x^4) + (\frac{h^2}{4}x^3 - \frac{h}{2}x^4 + \frac{1}{4}x^5 - nh^2x^2 + 2nhx^3 - nx^4) \right\} dx$$

$$= (\frac{1}{2}h^2x^2 - \frac{4}{3}hx^3 + x^4 + \frac{1}{3}h^2x^3 - \frac{3}{4}hx^4 + \frac{2}{5}x^5 + \frac{1}{16}h^2x^4 - \frac{1}{10}hx^5 + \frac{1}{24}x^6 - \frac{n}{3}h^2x^3 + \frac{n}{2}hx^4 - \frac{n}{5}x^5)\Big]_0^h$$

$$= h^4(\frac{1}{2} - \frac{4}{3} + 1) + h^5(\frac{1}{3} - \frac{3}{4} + \frac{2}{5}) + h^6(\frac{1}{16} - \frac{1}{10} + \frac{1}{24}) + h^5(-\frac{n}{3} + \frac{n}{2} - \frac{n}{5})$$

$$= h^4(\frac{1}{6} - \frac{(1+2n)}{60}h + \frac{1}{240}h^2) < 0$$

Since $h > 0$, $J\left[u,G\right] < 0$ implies that

$$\left[40 - 4(1 + 2n)\,h + h^2\right] < 0 \qquad (2.37)$$

or

$$(2 + 4n) - \sqrt{(2 + 4n)^2 - 40} < h < (2 + 4n) + \sqrt{(2 + 4n)^2 - 40} \qquad (2.38)$$

or

$$h > (2 + 4n) - \sqrt{(2 + 4n)^2 - 40} \qquad (2.39)$$

which gives a real value h for integer $n \geq 2$. Table IV compares the first and second zeros of several Laguerre polynomials for $n \geq 2$ with

the h from Eq (2.39). It can be seen from the table that, although the h occurs well above the first zero, it does fall between the first and second zeros in the examples shown.

Table IV
Zeros of Laguerre Polynomials

n	Polynomial $L_n(x)$	Interval with Zero	First Zero	Second Zero
2	$\frac{1}{2}x^2 - 2x + 1$	$[0,\ 2.254]$	0.586	3.414
3	$-\frac{1}{6}x^3 + \frac{3}{2}x^2 - 3x + 1$	$[0,\ 1.510]$	0.416	2.294
4	$\frac{1}{24}x^4 - \frac{2}{3}x^3 + 3x^2 - 4x + 1$	$[0,\ 1.148]$	0.323	1.746

(Ref 4:853)

2.3 Minimum Values of Certain Second Order Differential Equations

In addition to the proof of existence of a zero on an interval, it may be useful to determine the existence of a minimum value of a function $v(x)$ on an interval $[a,b]$, where $v(x)$ either comes "close" to or crosses the x axis. If the interval $[a,b]$ contains no zeros of $v(x)$, that is, either $v(x) > 0$ or $v(x) < 0$ for $a \leq x \leq b$, then the interval $[a,b]$ is said to be zero-free. Komkov derived the following theorem concerning zero-free intervals of a function $v(x)$ which satisfies a certain class of differential equations.

Theorem 2.3 - Komkov (Ref 13:26-28). If a function $v(x)$ satisfies the differential equation

$$L[v] = [A(x)\ v'(x)]' + C(x)\ f[v(x)] = 0 \tag{2.40}$$

subject to the following conditions:

(1) $f[v(x)] = [v(x)]^n$, $n > 1$;

(2) $A(x) > 0$ is differentiable and $C(x)$ is continuous on $[a,b]$;

(3) an admissible trial function $u(x)$ on $[a,b]$ has an associated differentiable function $G[u]$ such that $G[u(a)] = G[u(b)] = 0$ and $G[u(x)] > 0$ for $a < x < b$; and

(4) $J[u,G] < 0$ with

$$J[u,G] = \int_a^b \left\{ A(x)\,[u'(x)]^2 - C(x)\,G[u] \right\} dx \tag{2.41}$$

then there exists a value $m = \max\limits_{x \in [a,b]} \dfrac{g^2(u)}{4\,G[u]} > 0$, $g(u) = \dfrac{dG}{du}$, such that

$$v(x) < \left(\frac{m}{n}\right)^{\frac{1}{n-1}} \tag{2.42}$$

on some subinterval of $[a,b]$.

__Proof__. If $v(x) = 0$ at some point in $[a,b]$, then the theorem is immediately true. Therefore, if it is assumed that $[a,b]$ is zero-free, then $f(v) = [v(x)]^n \neq 0$ on $[a,b]$ and the following inequality may be derived:

$$0 \leq \int_a^b \left(\left[A\left(u' - \frac{gv'}{2f}\right)^2 + \frac{d}{dx}\left(\frac{A\,v'\,G}{f}\right) \right]\right) dx \tag{2.43}$$

$$= \int_a^b \left(\left[A(u')^2 - A\,u'\,\frac{gv'}{f} + A\,\frac{g^2(v')^2}{4f^2} \right] \right.$$

$$\left. + \left[\frac{f\left[(Av')'\,G + Av'\,gu'\right] - Av'\,G\left(\frac{df}{dv}\,v'\right)}{f^2} \right] \right) dx$$

$$= \int_a^b \left\{ A(u')^2 - A\,u' \frac{gv'}{f} + A \frac{g^2 (v')^2}{4f^2} \right.$$

$$\left. + \left[\frac{-(Cf)G + A\,v'\,gu'}{f} - \frac{A\,v'\,G\,(nv^{n-1})\,v'}{f^2} \right] \right\} dx$$

$$= \int_a^b \left\{ \left[A(u')^2 - C\,G\right] + \frac{A(v')^2}{4f^2} (g^2 - 4G\,nv^{n-1}) + \frac{A\,v'}{f} (u'g - gu') \right\} dx$$

$$= J\left[u,G\right] + \int_a^b \left[\frac{A(v')^2}{4f^2} (g^2 - 4G\,nv^{n-1}) \right] dx$$

Since $J\left[u,G\right] < 0$, this implies that $(g^2 - 4G\,nv^{n-1}) > 0$ for Eq (2.43) to be positive. This is equivalent to saying that, for some $\bar{x}$ in $[a,b]$,

$$\left[v(\bar{x})\right]^{n-1} < \frac{g\left[u(\bar{x})\right]}{4G\left[u(\bar{x})\right]\,n} \leq \max_{x \in [a,b]} \frac{g(u)}{4G\left[u\right]\,n} = \frac{m}{n} \qquad (2.44)$$

or

$$v(\bar{x}) < \left(\frac{m}{n}\right)^{\frac{1}{n-1}}$$

In the specific case where n is an odd integer, $\left[v(x)\right]^{n-1} > 0$ and Eq (2.42) becomes

$$\left|v(x)\right| < \left(\frac{m}{n}\right)^{\frac{1}{n-1}} \qquad (2.45)$$

Example 2.6. Emden's equation, which occurs in astrophysics, may be written as

$$\left[x^2 v'(x)\right]' + x^2 \left[v(x)\right]^n = 0 \qquad (2.46)$$

Whereas the full solution to this equation is unknown, Theorem 2.3 may be used to determine how close the function $v(x)$ will come to zero on $[a,b]$. For $h > 0$, a value p will be used to minimize h, where $G[u] = p\,u^2$, $u = x(h-x)$, and $J[u,G] < 0$:

$$J[u,G] = \int_0^h [A(u')^2 - C\,G]\,dx = \int_0^h \{x^2(h^2 - 4hx + 4x^2)$$

$$- x^2[x^2 p\,(h^2 - 2hx + x^2)]\}\,dx$$

$$= \int_0^h [\,h^2x^2 - 4hx^3 + 4x^4 - p(h^2x^4 - 2hx^5 + x^6)\,]\,dx$$

$$= \left(\frac{h^2}{3}x^3 - hx^4 + \frac{4}{5}x^5 - \frac{ph^2}{5}x^5 + \frac{ph}{6}x^6 - \frac{p}{7}x^7\right)\Big]_0^h$$

$$= h^5\left(\frac{1}{3} - 1 + \frac{4}{5}\right) - h^7\left(\frac{p}{5} - \frac{p}{6} + \frac{p}{7}\right)$$

$$= h^5\left(\frac{2}{15} - \frac{ph^2}{105}\right) < 0$$

This implies that $p > \frac{14}{h^2}$, and the value m becomes

$$m = \max_{x \in [0,h]} \frac{g^2}{4G} = \max_{x \in [0,h]} \frac{(2\,p\,u)^2}{4pu^2} = p \qquad (2.47)$$

Thus, in the specific case where $n = 5$ and $h = 2$, for any $p > \frac{14}{2^2} = 3.5$, $v(x)$ will attain a bound

$$|v\,(x)| < \left(\frac{m}{n}\right)^{\frac{1}{p-1}} = \left(\frac{p}{5}\right)^{\frac{1}{4}} \qquad (2.48)$$

on the interval $[0,2]$. For a $p = 3.51$, the result $|v(x)| < .916$ is

slightly better than Komkov's result $|v(x)| < .945$ for the same example (Ref 12:380-381). This slightly lower minimum value of v(x) arises from the use of a variable coefficient p with the functional $G[u]$.

Another minimum value theorem may be derived from Theorem 2.2 by introducing an intermediate term $2B(x)\, v'(x)$ into Eq (2.40). The same conditions (1-5) are used from Theorem 2.2, except that $\alpha(w) = 1$ and the expression $(g^2(u) - 4G[u] \frac{df}{dw})$ is not restricted to negative values.

<u>Theorem 2.4</u>. If the function v(x) satisfies the differential equation

$$L[v] = [A(x)\, v'(x)]' + 2B(x)\, v'(x) + C(x)\, f[v(x)] = 0 \qquad (2.49)$$

and if conditions (1-5) of Theorem 2.2 hold on an interval $[a,b]$, except that $g^2(u) - 4G[u] \frac{df}{dv} = K[u,f]$ may be positive and that $\alpha(v) = 1$, then for some $x_o \in [a,b]$ either (i) $f'[v(x_o)] = \frac{df}{dv(x_o)} < 1$ or (ii) $v(x_o) = 0$.

<u>Proof</u>. The proof parallels that of Theorem 2.2; assuming $v(x) \neq 0$ yields the following inequality:

$$0 \leq J[u,G] + \int_a^b \left[\frac{A(v')^2}{4f^2} (g^2 - 4f'G) \right] dx$$

$$+ \int_a^b \left[\frac{B\, v'\, G^{\frac{1}{2}}}{f} (g - 2G^{\frac{1}{2}}) \right] dx \qquad (2.50)$$

Since $(g - 2G^{\frac{1}{2}}) = 0$ by condition (4),

$$\int_a^b \left[\frac{A(v')^2}{f^2} (g^2 - 4f'\, G) \right] dx \leq 0$$

leads to a contradiction; this implies that $v(x_o) = 0$ for some $x_o \in [a,b]$. However, if

$$\int_a^b \left[\frac{A\,(v')^2}{f^2} (g^2 - 4f'\,G) \right] dx > 0$$

then $(g^2 - 4f'\,G) > 0$ for some $x_o \in [a,b]$, implying that $f' < \frac{g^2}{4G} = 1$. This completes the proof.

<u>Example 2.7</u>. A form of Duffing's equation,

$$v'' + p\,v' + (q^2 v + r v^3) = 0 \tag{2.51}$$

which arises in mechanics in the study of hard springs $(r > 0)$ and soft springs $(r < 0)$, may be investigated using Theorem 2.4 (Ref 14:16-18). With $Q(x) = \frac{p^2}{4}$, $A = 1$, $B = \frac{p}{2}$, $C = 1$, and $G = u^2$, and the trial function $u(x) = x(h-x)$ on the interval $[0,h]$, $J[u,G]$ from Eq (2.27) becomes

$$J[u,G] = \int_0^h \left\{ A(u')^2 - 2B\,G^{\frac{1}{2}}\,u' + (Q - C)\,G \right\} dx \tag{2.52}$$

$$= \int_0^h \left[1(h - 2x)^2 - p(xh-x^2)(h-2x) + \left(\frac{p^2}{4} - 1\right)(xh-x^2)^2 \right] dx$$

$$= \int_0^h [\, h^2 - 4hx + 4x^2 - ph^2 x + 3phx^2 - 2px^3$$

$$+ \left(\frac{p^2}{4} - 1\right) (h^2x^2 - 2hx^3 + x^4) \,]\, dx$$

$$= h^2 x - 2hx^2 + \frac{4}{3} x^3 - \frac{p}{2} h^2 x^2 + phx^3 - \frac{p}{2} x^4$$

$$+ \left(\frac{p^2}{4} - 1\right) \left(\frac{h^2}{3} x^3 - \frac{h}{2} x^4 + \frac{1}{5} x^5\right) \Big]_0^h$$

$$= (h^3 - 2h^3 + \frac{4}{3} h^3) + (- \frac{p}{2} h^4 + ph^4 - \frac{p}{2} h^4)$$

$$+ (\frac{p^2}{4} - 1) (\frac{1}{3} h^5 - \frac{1}{2} h^5 + \frac{1}{5} h^5)$$

$$= h^3 [\frac{1}{3} + \frac{h^2}{30} (\frac{p^2}{4} - 1)] < 0$$

This implies that, for $|p| < 2$, $h > [\frac{40}{(4-p^2)}]^{\frac{1}{2}}$; the coefficient p determines the size of the interval $[0,h]$ under consideration. If $|p| \geq 2$, the differential equation Eq (2.51) may be multiplied by a constant ϵ to make $|\epsilon p| < 0$. This will change the coefficients A, B, and C in $L[v]$ in Eq (2.49).

After the size of the interval $[0,h]$ is determined, the derivative

$$f'(v) = \frac{df}{dv} = \frac{d}{dv} (q^2 v + rv^3) = q^2 + 3 rv^2 \qquad (2.53)$$

is examined to determine if $v(x) = 0$ or if $v(x)$ comes close to zero. For example, if $p = 1$, then $h > (\frac{40}{3})^{\frac{1}{2}} \approx 3.652$ and the interval under examination is $[0, 3.652]$. If $q = r = \frac{1}{2}$, then $f' = \frac{1}{4} + \frac{3}{2} v^2(x) < 1$ implies that $v^2(x) < \frac{1}{2}$, or $|v(x)| < .7071$ for some $x \in [0, 3.652]$. However, if $q = r = 1$ in Eq (2.51), then it is always the case that $f' = 1 + v^2(x) \geq 1$ since $v^2(x) \geq 0$. This means that $g^2 - 4f'G \leq 0$ in Eq (2.50), which implies that $v(x) = 0$ for some $x \in [0, 3.652]$. Thus, Theorem 2.4 can provide a bounded value, and in some cases a zero, of a function $v(x)$ which satisfies the differential equation $L[v] = 0$ in Eq (2.49).

The above theorem is also applicable to the equation

$$u''(t) + (u - ku^2) = 0 \qquad (2.54)$$

which is used in the theory of equatorial satellite orbits of an oblate spheroid, where u represents the variation in the radius of orbit and t represents the angular variable (Ref 14:23-24).

III. Zeros of Real-Valued Functions of Two Real Variables

In this chapter sufficient conditions are established for the existence of a real zero (or a real minimum value) of a continuous real-valued function $f(x_1,x_2)$ with two real variables within a region R. The theorems are presented for only two variables; however, they can be extended to apply to functions of several variables.

The real zeros of $f(x_1,x_2)$ will be considered within the region R defined by the rectangle $R = \{(x_1,x_2): 0 \le x_1 \le h,\ 0 \le x_2 \le k\}$; for other regions, basic transformations may be applied as in the previous chapter.

The function $f(x_1,x_2)$ is required to be a solution of the differential equation

$$L[f] = \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial}{\partial x_i}\left[a_{ij}(x_1,x_2)\frac{\partial f}{\partial x_j}\right] + C(x_1,x_2)[f(x_1,x_2)]^{2s-1} = 0 \tag{3.1}$$

where s is an integer, $s \ge 1$, $C(x_1,x_2)$ is continuous on R, and the elements $a_{ij}(x_1,x_2)$ are differentiable on R and form a symmetric, positive semi-definite matrix

$$A(x_1,x_2) = \begin{bmatrix} a_{11}(x_1,x_2) & a_{12}(x_1,x_2) \\ a_{21}(x_1,x_2) & a_{22}(x_1,x_2) \end{bmatrix} \tag{3.2}$$

Admissible trial functions $u(x_1,x_2)$ will be used which are differentiable on R, zero on the boundary of R, and positive in the interior of R.

Two such trial functions for the R defined above are

$$u(x_1,x_2) = x_1 (h-x_1) x_2 (k-x_2) \tag{3.3}$$

and

$$u(x_1,x_2) = \sin \left(\frac{\pi x_1}{h} \right) \sin \left(\frac{\pi x_2}{k} \right) \tag{3.4}$$

These trial functions vanish on the boundary of R and are positive within R.

Associated with $f(x_1,x_2)$ and the differential equation $L[f] = 0$ will be the functional $J[u]$ which is defined, in its most basic form, by

$$J[u] = \iint_R \Big\{ s \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij}(x_1,x_2) \left(\frac{\partial u}{\partial x_i} \right) \left(\frac{\partial u}{\partial x_j} \right)$$

$$- C(x_1,x_2)\, u^{2s}(x_1,x_2) \Big\}\, dx_1 dx_2 \tag{3.5}$$

As with $J[u]$ in the single variable case, whenever $J[u] < 0$, there exists at least one zero of $f(x_1,x_2)$ within the closed region R for $s = 1$, or there exists under certain conditions a minimum value of $f(x_1,x_2)$ within R for $s > 1$.

The following lemma, which will be used to prove theorems later in this chapter, establishes the equality

$$E[u,f] = d(J[u]) \tag{3.6}$$

where $d(J[u])$ is defined to be the integrand of $J[u]$ in Eq (3.5) and $E[u,f]$ is an energy functional defined below on a region R in which $f(x_1,x_2) \neq 0$.

__Lemma__. If, within a given region R, $f(x_1,x_2)$ is a function which satisfies Eq (3.1) and does not vanish, $u(x_1,x_2)$ is an admissible trial function, the integer $s \geq 1$, and the coefficients $a_{ij}(x_1,x_2)$ form the positive semi-definite matrix shown in Eq (3.2), then the following identity holds:

$$d(J[u]) = E[u,f] = s\left[\frac{u^{2s-2}}{f^{2s-4}}\right] H\left(\frac{u}{f}\right) + (s-1)\left(\frac{u}{f}\right)^{2s} H(f)$$

$$+ s\left[1 - \left(\frac{u}{f}\right)^{2s-2}\right] H(u)$$

$$+ \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial}{\partial x_i}\left[\frac{u^{2s}}{f^{2s-1}} a_{ij} \frac{\partial f}{\partial x_j}\right] \tag{3.7}$$

where $d(J[u])$ represents the integrand of $J[u]$ in Eq (3.5) and $H(*)$ is defined by

$$H(*) = \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} \left(\frac{\partial *}{\partial x_i}\right) \left(\frac{\partial *}{\partial x_j}\right) \tag{3.8}$$

__Proof__. Using the fact that $a_{12} = a_{21}$ in the symmetric matrix A, the expansion of $E[u,f]$ and cancellation of terms produces the desired identity:

$$E[u,f] = s\left[\frac{u^{2s-2}}{f^{2s-4}}\right] H\left(\frac{u}{f}\right) + (s-1)\left(\frac{u}{f}\right)^{2s} H(f) + s\left[1 - \left(\frac{u}{f}\right)^{2s-2}\right] H(u)$$

$$+ \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial}{\partial x_i}\left[\frac{u^{2s}}{f^{2s-1}} a_{ij} \frac{\partial f}{\partial x_j}\right] \tag{3.9}$$

$$- \; s \, a_{11} \left[\frac{u^{2s-2}}{f^{2s-4}} \right] \left[\frac{f \frac{\partial u}{\partial x_1} - u \frac{\partial f}{\partial x_1}}{f^2} \right]^2$$

$$+ \; s \, a_{12} \left[\frac{u^{2s-2}}{f^{2s-4}} \right] \left[\frac{f \frac{\partial u}{\partial x_1} - u \frac{\partial f}{\partial x_1}}{f^2} \right] \left[\frac{f \frac{\partial u}{\partial x_2} - u \frac{\partial f}{\partial x_2}}{f^2} \right]$$

$$+ \; s \, a_{21} \left[\frac{u^{2s-2}}{f^{2s-4}} \right] \left[\frac{f \frac{\partial u}{\partial x_1} - u \frac{\partial f}{\partial x_1}}{f^2} \right] \left[\frac{f \frac{\partial u}{\partial x_2} - u \frac{\partial f}{\partial x_2}}{f^2} \right]$$

$$+ \; s \, a_{22} \left[\frac{u^{2s-2}}{f^{2s-4}} \right] \left[\frac{f \frac{\partial u}{\partial x_2} - u \frac{\partial f}{\partial x_2}}{f^2} \right]^2$$

$$+ \left\{ (s-1) \sum_{i=1}^{2} \sum_{j=1}^{2} \left(\frac{u}{f} \right)^{2s} \left(\frac{\partial f}{\partial x_i} \right) \left(\frac{\partial f}{\partial x_j} \right) \right\}$$

$$+ \left\{ s \left[1 - \left(\frac{u}{f} \right)^{2s-2} \right] \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} \left(\frac{\partial u}{\partial x_i} \right) \left(\frac{\partial u}{\partial x_j} \right) \right\}$$

$$+ \left\{ \sum_{i=1}^{2} \sum_{j=1}^{2} \left(\frac{u^{2s}}{f^{2s-1}} \right) \left[\frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial f}{\partial x_j} \right) \right] \right.$$

$$\left. + \sum_{i=1}^{2} \sum_{j=1}^{2} \left(a_{ij} \frac{\partial f}{\partial x_j} \right) \left[\frac{\partial}{\partial x_i} \left(\frac{u^{2s}}{f^{2s-1}} \right) \right] \right\}$$

$$= s\, a_{11} \left[\frac{u^{2s-2}}{f^{2s-4}} \right] \left[\frac{f^2 \left(\frac{\partial u}{\partial x_1} \right)^2 - 2uf \left(\frac{\partial u}{\partial x_1} \right)\left(\frac{\partial f}{\partial x_1} \right) + u^2 \left(\frac{\partial f}{\partial x_1} \right)^2}{f^4} \right]$$

$$+ s\, a_{12} \left[\frac{u^{2s-2}}{f^{2s-4}} \right] \left[\frac{f^2 \left(\frac{\partial u}{\partial x_1}\right)\left(\frac{\partial u}{\partial x_2}\right) - uf \left(\frac{\partial f}{\partial x_1}\right)\left(\frac{\partial u}{\partial x_2}\right) - uf \left(\frac{\partial u}{\partial x_1}\right)\left(\frac{\partial f}{\partial x_2}\right) + u^2 \left(\frac{\partial f}{\partial x_1}\right)\left(\frac{\partial f}{\partial x_2}\right)}{f^4} \right]$$

$$+ s\, a_{21} \left[\frac{u^{2s-2}}{f^{2s-4}} \right] \left[\frac{f^2 \left(\frac{\partial u}{\partial x_1}\right)\left(\frac{\partial u}{\partial x_2}\right) - uf \left(\frac{\partial f}{\partial x_1}\right)\left(\frac{\partial u}{\partial x_2}\right) - uf \left(\frac{\partial u}{\partial x_1}\right)\left(\frac{\partial u}{\partial x_2}\right) + u^2 \left(\frac{\partial f}{\partial x_1}\right)\left(\frac{\partial f}{\partial x_2}\right)}{f^4} \right]$$

$$+ s\, a_{22} \left[\frac{u^{2s-2}}{f^{2s-4}} \right] \left[\frac{f^2 \left(\frac{\partial u}{\partial x_2} \right)^2 - 2uf \left(\frac{\partial u}{\partial x_2} \right)\left(\frac{\partial f}{\partial x_2} \right) + u^2 \left(\frac{\partial f}{\partial x_2} \right)^2}{f^4} \right]$$

$$+ \left\{ \sum_{i=1}^{2} \sum_{j=1}^{2} \left(\frac{u^{2s}}{f^{2s-1}} \right) \left[\frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial f}{\partial x_j} \right) \right] \right.$$

$$\left. + \sum_{i=1}^{2} \sum_{j=1}^{2} \left(a_{ij} \frac{\partial f}{\partial x_j} \right) \left[\frac{\partial}{\partial x_i} \left(\frac{u^{2s}}{f^{2s-1}} \right) \right] \right\}$$

$$+ \left\{ (s-1) \sum_{i=1}^{2} \sum_{j=1}^{2} \left(\frac{u}{f} \right)^{2s} \left(\frac{\partial f}{\partial x_i} \right)\left(\frac{\partial f}{\partial x_j} \right) \right\}$$

$$+ \left\{ s\left[1 - \left(\frac{u}{f} \right)^{2s-2} \right] \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} \left(\frac{\partial u}{\partial x_i} \right)\left(\frac{\partial u}{\partial x_j} \right) \right\}$$

$$= \{ s\, a_{11}(\tfrac{u}{f})^{2s-2}(\tfrac{\partial u}{\partial x_1})^2 - 2s\, a_{11}(\tfrac{u}{f})^{2s-1}(\tfrac{\partial u}{\partial x_1})(\tfrac{\partial f}{\partial x_1}) + s a_{11}(\tfrac{u}{f})^{2s}(\tfrac{\partial f}{\partial x_1})^2$$

$$+ s\, a_{12}(\tfrac{u}{f})^{2s-2}(\tfrac{\partial u}{\partial x_1})(\tfrac{\partial u}{\partial x_2}) - s\, a_{12}(\tfrac{u}{f})^{2s-1}(\tfrac{\partial f}{\partial x_1})(\tfrac{\partial u}{\partial x_2})$$

$$- s\, a_{12}(\tfrac{u}{f})^{2s-1}(\tfrac{\partial u}{\partial x_1})(\tfrac{\partial f}{\partial x_2}) + s\, a_{12}(\tfrac{u}{f})^{2s}(\tfrac{\partial f}{\partial x_1})(\tfrac{\partial f}{\partial x_2})$$

$$+ s\, a_{21}(\tfrac{u}{f})^{2s-2}(\tfrac{\partial u}{\partial x_1})(\tfrac{\partial u}{\partial x_2}) - s\, a_{21}(\tfrac{u}{f})^{2s-1}(\tfrac{\partial f}{\partial x_1})(\tfrac{\partial u}{\partial x_2})$$

$$- s\, a_{21}(\tfrac{u}{f})^{2s-1}(\tfrac{\partial u}{\partial x_1})(\tfrac{\partial f}{\partial x_2}) + s\, a_{21}(\tfrac{u}{f})^{2s}(\tfrac{\partial f}{\partial x_1})(\tfrac{\partial f}{\partial x_2})$$

$$+ s\, a_{22}(\tfrac{u}{f})^{2s-2}(\tfrac{\partial u}{\partial x_2})^2 - 2s\, a_{22}(\tfrac{u}{f})^{2s-1}(\tfrac{\partial u}{\partial x_2})(\tfrac{\partial f}{\partial x_2})$$

$$+ s\, a_{22}(\tfrac{u}{f})^{2s}(\tfrac{\partial f}{\partial x_2})^2 \}$$

$$+ \{ (\frac{u^{2s}}{f^{2s-1}}) \sum_{i=1}^{2} \sum_{j=1}^{2} [\frac{\partial}{\partial x_i}(a_{ij} \frac{\partial f}{\partial x_j})] + 2s a_{11}(\tfrac{u}{f})^{2s-1}(\tfrac{\partial u}{\partial x_1})(\tfrac{\partial f}{\partial x_1})$$

$$- (2s-1)\, a_{11}(\tfrac{u}{f})^{2s}(\tfrac{\partial f}{\partial x_1})^2 + 2s\, a_{12}(\tfrac{u}{f})^{2s-1}(\tfrac{\partial u}{\partial x_1})(\tfrac{\partial f}{\partial x_2})$$

$$- (2s-1)\, a_{12}(\tfrac{u}{f})^{2s}(\tfrac{\partial f}{\partial x_1})(\tfrac{\partial f}{\partial x_2}) + 2s\, a_{21}(\tfrac{u}{f})^{2s-1}(\tfrac{\partial f}{\partial x_1})(\tfrac{\partial u}{\partial x_2})$$

$$- (2s-1)\, a_{21}(\tfrac{u}{f})^{2s}(\tfrac{\partial f}{\partial x_1})(\tfrac{\partial f}{\partial x_2}) + 2s\, a_{22}(\tfrac{u}{f})^{2s-1}(\tfrac{\partial u}{\partial x_2})(\tfrac{\partial f}{\partial x_2})$$

$$- (2s-1)\ a_{22}\ (\frac{u}{f})^{2s}(\frac{\partial f}{\partial x_2})^2 \} + \{ (s-1)\ a_{11}(\frac{u}{f})^{2s}(\frac{\partial f}{\partial x_1})^2$$

$$+ (s-1)\ a_{12}(\frac{u}{f})^{2s}(\frac{\partial f}{\partial x_1})(\frac{\partial f}{\partial x_2}) + (s-1)\ a_{21}(\frac{u}{f})^{2s}(\frac{\partial f}{\partial x_1})(\frac{\partial f}{\partial x_2})$$

$$+ (s-1)\ a_{22}(\frac{u}{f})^{2s}(\frac{\partial f}{\partial x_2})^2 \} + \{ s \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij}(\frac{\partial u}{\partial x_i})(\frac{\partial u}{\partial x_j})$$

$$- s\ a_{11}(\frac{u}{f})^{2s-2}(\frac{\partial u}{\partial x_1})^2 - s\ a_{12}(\frac{u}{f})^{2s-2}(\frac{\partial u}{\partial x_1})(\frac{\partial u}{\partial x_2})$$

$$- s\ a_{21}(\frac{u}{f})^{2s-2}(\frac{\partial u}{\partial x_1})(\frac{\partial u}{\partial x_2}) - s\ a_{22}(\frac{u}{f})^{2s-2}(\frac{\partial u}{\partial x_2})^2 \}$$

$$= (\frac{u^{2s}}{f^{2s-1}}) \sum_{i=1}^{2} \sum_{j=1}^{2} [\frac{\partial}{\partial x_i}\ (a_{ij}\ \frac{\partial f}{\partial x_j})]$$

$$+ s \sum_{i=1}^{2} \sum_{j=1}^{2} [a_{ij}\ (\frac{\partial u}{\partial x_i})(\frac{\partial u}{\partial x_j})] + [0]$$

$$= (\frac{u^{2s}}{f^{2s-1}}) \sum_{i=1}^{2} \sum_{j=1}^{2} [\frac{\partial}{\partial x_i}\ (a_{ij}\ \frac{\partial f}{\partial x_j})]$$

$$+ s \sum_{i=1}^{2} \sum_{j=1}^{2} [a_{ij}(\frac{\partial u}{\partial x_i})(\frac{\partial u}{\partial x_j})] + [(\frac{u^{2s}}{f^{2s-1}})\ c\ f^{2s-1} - u^{2s}\ c]$$

$$= \left(\frac{u^{2s}}{f^{2s-1}}\right) \left\{ \sum_{i=1}^{2} \sum_{j=1}^{2} \left[\frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial f}{\partial x_j}\right)\right] + C\, f^{2s-1} \right\}$$

$$+ \left\{ s \sum_{i=1}^{2} \sum_{j=1}^{2} \left[a_{ij} \left(\frac{\partial u}{\partial x_i}\right)\left(\frac{\partial u}{\partial x_j}\right)\right] - C\, u^{2s} \right\}$$

$$= \left(\frac{u^{2s}}{f^{2s-1}}\right) L[f] + d(J[u]) = d(J[u])$$

Thus $E[u,f] = d(J[u])$.

If a region R can be found for which a trial function $u(x_1,x_2)$ makes $J[u] < 0$, then $f(x_1,x_2)$ will either vanish somewhere in R or attain some minimum value on R, depending on the value of s. For $s = 1$, the integration of Eq (3.7) gives

$$J[u] = \int_R\int E[u,f]\, dx_1 dx_2 = \int_R\int \left\{ f^2\, H\left(\frac{u}{f}\right) \right\} dx_1 dx_2$$

$$+ \int_R\int \left\{ \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial}{\partial x_i} \left[\frac{u^2}{f}\, a_{ij} \left(\frac{\partial f}{\partial x_j}\right)\right] \right\} dx_1 dx_2 \qquad (3.10)$$

The result of the first integration is non-negative, since the positive semi-definite matrix A makes H(*) in Eq (3.8) non-negative. The result of the second integration is zero, since the trial function $u(x_1,x_2) = 0$ on the boundary of R. This leads to the following theorem:

<u>Theorem 3.1</u>. If $f(x_1,x_2)$ is a function which satisfies the differential equation $L[f] = 0$ in Eq (3.1) with $s = 1$, and if for some region R

$$J[u] = \int_R\int \left\{ \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} \left(\frac{\partial u}{\partial x_i}\right)\left(\frac{\partial u}{\partial x_j}\right) - C\, u \right\} dx_1 dx_2 < 0 \qquad (3.11)$$

then $f(x_1,x_2) = 0$ in some subregion of R.

__Proof__. If it is assumed that $f(x_1,x_2) \neq 0$ on R, then $E[u,f]$ is well-defined on R. The combination of Eqs (3.10) and (3.11) gives rise to the contradiction

$$0 \leq \int_R\int E[u,f]\, dx_1 dx_2 = J[u] < 0$$

Therefore, the assumption that $f(x_1,x_2) \neq 0$ on R is false, and the proof is complete.

__Example 3.1__. A Laplace equation of the form

$$f_{xx} + f_{yy} + f(x,y) = 0 \tag{3.12}$$

matches Eq (3.1), with $a_{11} = a_{22} = 1$ and $a_{12} = a_{21} = 0$ making the matrix A positive semi-definite and with $C = 1$ and $s = 1$. The trial function $u(x,y) = x(h-x)\, y\,(k-y)$ fits a rectangular region $R = \{(x,y): 0 \leq x \leq h,\ 0 \leq y \leq k\}$. To determine values for h and k such that the region R contains a zero, h and k are expanded until the functional $J[u] < 0$. Thus

$$J[u] = \int_0^k \int_0^h \{ 1 (\frac{\partial u}{\partial x})^2 + 1 (\frac{\partial u}{\partial y})^2 - 1 (u)^2 \}\, dxdy$$

$$= \int_0^k \int_0^h \{ [(h-2x)(ky-y^2)]^2 + [(hx-x^2)(k-2y)]^2 - [(hx-x^2)(ky-y^2)]^2 \}\, dxdy$$

$$= \int_0^k \{ (ky-y^2)^2 \int_0^h (h^2 - 4hx + 4x^2 - h^2x^2 + 2hx^3 - x^4)\, dx$$

$$+ (k-2y)^2 \int_0^h (h^2x^2 - 2hx^3 + x^4)\, dx \}\, dy$$

$$= \int_0^k \{ (ky-y^2)^2 \; (h^2x - 2hx^2 + \frac{4-h^2}{3} x^3 + \frac{h}{2} x^4 - \frac{1}{5} x^5)\Big]_0^h$$

$$+ (k-2y)^2 \; (\frac{h^2}{3} x^3 - \frac{h}{2} x^4 + \frac{1}{5} x^5)\Big]_0^h \} \, dy$$

$$= \int_0^k \{ (ky-y^2)^2 \; (h^3 - 2h^3 + \frac{4}{3} h^3 - \frac{1}{3} h^5 + \frac{1}{2} h^5 - \frac{1}{5} h^5)$$

$$+ (k-2y)^2 \; (\frac{1}{3} h^5 - \frac{1}{2} h^5 + \frac{1}{5} h^5) \} \, dy$$

$$= (\frac{1}{3} h^3 - \frac{1}{30} h^5) \int_0^k (k^2y^2 - 2ky^3 + y^4) \, dy$$

$$+ (\frac{1}{30} h^5) \int_0^k (k^2 - 4ky + 4y^2) \, dy$$

$$= (\frac{1}{3} h^3 - \frac{1}{30} h^5) \frac{1}{30} k^5 + (\frac{1}{30} h^5) \frac{1}{3} k^3$$

$$= \frac{h^3k^3}{900} (10h^2 - h^2k^2 + 10k^2) < 0$$

Therefore, for positive h and k values, $(10h^2 - h^2k^2 + 10k^2) < 0$; this implies that for a fixed value h

$$k > \left(\frac{10h^2}{h^2-10} \right)^{\frac{1}{2}} \qquad (3.13)$$

For real solutions, h must be greater than $\sqrt{10}$; also, the symmetry of the functional $L[f]$ implies that $k > \sqrt{10}$. In the instance where $h = k$, a square region R results, and the solution of Eq (3.13) gives $h > 2\sqrt{5} \approx 4.472$. Thus, every solution to the Laplace equation (3.11) must have a zero on $R = \{ (x,y): 0 \leq x \leq 4.472, \; 0 \leq y \leq 4.472 \}$.

Next, Eq (3.1) is investigated for integer values of $s > 1$, and a minimum value is determined for a region R.

<u>Theorem 3.2</u>. If $f(x_1,x_2)$ is a function which satisfies $L[f] = 0$ in Eq (3.1) with the integer $s > 1$, and if, for some region R, the functional $J[u]$ in Eq (3.5) is negative, then

$$|f(\bar{x}_1,\bar{x}_2)| < \max_{(x_1,x_2)\in R} u(x_1,x_2)$$

for some point $(\bar{x}_1,\bar{x}_2) \in R$.

<u>Proof</u>. If $f(x_1,x_2) = 0$ somewhere in R, the proof is trivial, since the trial function $u(x_1,x_2) > 0$ in the interior of R. If $f(x_1,x_2) \neq 0$ in R, then $E[u,f]$ is well-defined in R, and integration of Eq (3.7) produces the inequality

$$0 > J[u] = \iint_R E[u,f]\, dx_1 dx_2 = \iint_R s\left[\frac{u^{2s-2}}{f^{2s-4}}\right] H\left(\frac{u}{f}\right) dx_1 dx_2$$

$$+ \iint_R (s-1)\left(\frac{u}{f}\right)^{2s} H(f)\, dx_1 dx_2$$

$$+ \iint_R s\left[1 - \left(\frac{u}{f}\right)^{2s-2}\right] H(u)\, dx_1 dx_2$$

$$+ \iint_R \left[\sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial}{\partial x_i}\left(\frac{u^{2s}}{f^{2s-1}} a_{ij} \frac{\partial f}{\partial x_j}\right)\right] dx_1 dx_2 \tag{3.14}$$

The first and second integrals in Eq (3.14) are non-negative since the integrands are non-negative. The fourth integral is zero since $u = 0$ on the boundaries of R. Thus, for $J[u]$ to be negative, the third integral must be negative; this implies that $\left[1 - \left(\frac{u}{f}\right)^{2s-2}\right] < 0$ in some subregion of R.

Therefore,

$$\left(\frac{u}{f}\right)^{2s-2} > 1 \quad \text{and} \quad \left|f(\bar{x}_1,\bar{x}_2)\right| < \max_{(x,y)\in R} u(x_1,x_2)$$

for some point $(\bar{x}_1,\bar{x}_2) \in R$, and the proof is complete.

<u>Example 3.2</u>. A non-linear Laplace equation

$$f_{xx} + f_{yy} + [f(x,y)]^3 = 0 \qquad (3.15)$$

fits the differential equation $L[f] = 0$ in Eq (3.1) with $a_{11} = a_{22} = C = 1$ and $a_{12} = a_{21} = 0$ but with $s = 2$. Eq (3.5) and the calculations of Example 3.1 are used to determine an R for which $J[u] < 0$:

$$J[u] = s\left[\frac{1}{90}(h^3k^5 + h^5k^3)\right] - \int_0^k\int_0^h (1\ u^{2s})dxdy$$

$$= \frac{1}{45}(h^3k^5 + h^5k^3) - \int_0^k\int_0^h [(hx-x^2)(ky-y^2)]^4 \, dxdy$$

$$= \frac{1}{45}(h^3k^5 + h^5k^3) - \int_0^k (ky-y^2)^4\left[\int_0^h (h^4x^4-4h^3x^5+6h^2x^6-4hx^7+x^8)dx\right]dy$$

$$= \frac{1}{45}(h^3k^5 + h^5k^3) - \int_0^k (ky-y^2)^4 \left\{ \left(\frac{1}{5}h^4x^5-\frac{2}{3}h^3x^6+\frac{6}{7}h^2x^7-\frac{1}{2}hx^8+\frac{1}{9}x^9\right)\Big]_0^h \right\} dy$$

$$= \frac{1}{45}(h^3k^5 + h^5k^3) - \frac{h^9}{630}\int_0^k (k^4y^4 - 4k^3y^5 + 6k^2y^6 - 4ky^7 + y^8)\,dy$$

$$= \frac{1}{45}(h^3k^5 + h^5k^3) - \frac{h^9}{630}\,\frac{k^9}{630} = \frac{h^3k^3}{396900}(8820h^2 - h^6k^6 + 8820k^2) < 0$$

If $h = k$, a square region R results, and $(8820h^2 - h^6k^6 + 8820k^2) < 0$ becomes $h^{10} > 17640$ or $h > (17640)^{1/10} \approx 2.66$. In this square region

$R = \{(x,y): 0 \leq x \leq 2.66, 0 \leq y \leq 2.66\}$, the maximum value of $u(x,y)$ will occur at the central point (1.33,1.33), where $u(1.33,1.33) \approx 3.13$. Thus, for some point $(\bar{x},\bar{y}) \in R$, $|f(\bar{x},\bar{y})| < \max\limits_{(x,y)\in R} u(x,y) \approx 3.13$.

These results are achieved using only the trial function $u(x,y) = (xh-x^2)(yk-y^2)$. In both the one-variable case presented in Chapter II and in the two-variable case in this chapter, the use of different trial functions may yield smaller regions containing zeros or minimums. However, the added complexity of the calculations often offsets the added accuracy that may occur.

IV. Zeros of Real Vector-Valued Functions

In optimization theory, a knowledge of zeros of the derivatives of vector-valued functions is needed in order to maximize or minimize those functions subject to constraints. In this chapter, sufficient conditions are established for the existence of a real zero of a vector-valued function

$$\vec{v}(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \\ \cdot \\ \cdot \\ \cdot \\ v_n(t) \end{pmatrix} = (v_1, v_2, \ldots, v_n)^T \tag{4.1}$$

in a finite-dimensional vector space. For the simplification of notation the letters u, v, w, and y will represent vectors $\vec{u}(t)$, $\vec{v}(t)$, $\vec{w}(t)$ and $\vec{y}(t)$ of a single real variable t, A and C will represent continuous functions A(t) and C(t), R will represent a constant symmetric positive-definite n-by-n matrix, and other letters will represent constants.

The notion of an inner product will be used extensively in this chapter. The inner product of two real n-dimensional column vectors v and w shall be defined by

$$\begin{aligned} < v,w > &= <(v_1,v_2,\ldots,v_n)^T, (w_1,w_2,\ldots,w_n)^T > \\ &= v^T w = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n \end{aligned} \tag{4.2}$$

Using this definition, three basic properties for inner products follow (Ref 15:6-7):

(1) $< v,w > = < w,v >$; (4.3)

(2) $< c_1v + c_2w, y > = c_1 < v,y > + c_2 < w,y >$; and

(3) $< v,v > \; > 0$ if $v \neq 0$, and $< v,v > = 0$ only if $v = 0$.

The following lemmas establish basic inner product identities to be used later.

__Lemma 4.1__. For two differentiable vector-valued functions v and w,

$$< v,w >' = \frac{d}{dt} (< v,w >) = < v,w' > + < v',w > \tag{4.4}$$

__Proof__. This is easily established by using the definition of an inner product from Eq (4.2):

$$\begin{aligned} < v,w >' &= (v_1w_1 + v_2w_2 + \ldots + v_nw_n)' \\ &= (v_1w_1' + v_1'w_1 + v_2w_2' + v_2'w_2 + \ldots + v_nw_n' + v_n'w_n) \\ &= (v_1w_1' + v_2w_2' + \ldots + v_nw_n') \\ &\qquad + (v_1'w_1 + v_2'w_2 + \ldots + v_n'w_n) \\ &= < v,w' > + < v',w > \end{aligned}$$

__Lemma 4.2__. If R is a constant symmetric matrix and v is a vector function, then $< v,Rv >' = 2 < v,Rv' >$.

__Proof__. Since R is constant and symmetric and the properties in Eq (4.3) apply, the following identity holds:

$$\begin{aligned} < v',Rv > &= (v')^T (Rv) = [(v')^TR]v = [R^T(v')]^Tv = [R(v')]^Tv \\ &= < Rv',v > = < v,Rv' > \end{aligned} \tag{4.5}$$

Thus

$$< v,Rv >' = < v,Rv'> + < v',Rv >$$
$$= < v,Rv'> + < v,Rv' >$$
$$= 2 < v,Rv'> \qquad (4.6)$$

To locate the zeros of a vector-valued function v(t), it will be required that v satisfies the inner product differential equation L [v] of the type

$$L[v] = [A < v,Rv >']' + C < v,Rv > = 0 \qquad (4.7)$$

where A(t) is differentiable, A(t) > 0, and C(t) is continuous on an interval [a,b]. A vector-valued admissible trial function u(t) will be used, in a manner similar to the trial functions of previous chapters, with the conditions that u(t) is differentiable on [a,b], u(a) = u(b) = 0, and u(t) > 0 for $t \in (a,b)$. Associated with the vector u will be a functional J [u], where

$$J[u] = \int_a^b [A(< u,Ru >')^2 - C(< u,Ru >)^2]\, dt \qquad (4.8)$$

The following theorem demonstrates that, whenever J [u] in Eq (4.8) can be made negative by varying the limits of integration a and b, v has at least one zero on [a,b].

<u>Theorem 4.1</u>. If v(t) is a vector which satisfies Eq (4.7), and if u(t) is a vector-valued admissible trial function on [a,b] such that J [u] < 0 for the J [u] defined in Eq (4.8), then v(t) has at least one zero on [a,b].

Proof. The proof by contradiction is done in much the same way as is the proof of Theorem 2.1. Since R is a positive-definite matrix, $< v,Rv > \geq 0$ by definition. If it is assumed that $< v,Rv > \not\equiv 0$ on $[a,b]$, then the above lemmas may be used to produce the following contradiction:

$$0 = \int_a^b \left[\left(\frac{< u,Ru >^2 A < v,Rv >'}{< v,Rv >} \right)'\right] dt \tag{4.9}$$

$$\leq \int_a^b \left\{\left(\frac{< u,Ru >^2 A < v,Rv >'}{< v,Rv >} \right)' + A\left[\left(\frac{< u,Ru >}{< v,Rv >} \right)'\right]^2 < v,Rv >^2 \right\} dt$$

$$= \int_a^b \left[\left(\frac{< u,Ru >^2 < v,Rv >'}{< v,Rv >} \right)'\right.$$

$$\left. + A\left(\frac{- < v,Rv >' < u,Ru > + < u,Ru >' < v,Rv >}{< v,Rv >} \right)^2 \right] dt$$

$$= \int_a^b \left[\left(\frac{< u,Ru >^2 < v,Rv >'}{< v,Rv >} \right)' + A\left(\frac{< v,Rv >' < u,Ru >}{< v,Rv >} \right)^2\right.$$

$$- 2A\left(\frac{< u,Ru > < u,Ru >' < v,Rv > < v,Rv >'}{< v,Rv >^2} \right)$$

$$\left. + A\left(< u,Ru >' \right)^2 \right] dt$$

$$= \int_a^b \left[\left(\frac{< u,Ru >^2 A < v,Rv >'}{< v,Rv >} \right)' + A\left(\frac{< v,Rv >' < u,Ru >}{< v,Rv >} \right)^2\right.$$

$$\left. - 2A\left(\frac{< u,Ru > < u,Ru >' < v,Rv >'}{< v,Rv >} \right) + A\left(< u,Ru >' \right)^2 \right] dt$$

$$= \int_a^b \left[\left(\frac{< u,Ru >^2 A < v,Rv >'}{< v,Rv >} \right)' + A\left(\frac{< v,Rv >' < u,Ru >}{< v,Rv >} \right)^2\right.$$

$$\left. - A\left(\frac{2 < u,Ru > < u,Ru >' < v,Rv >'}{< v,Rv >} \right) + A\left(< u,Ru >' \right)^2 \right] dt$$

$$= \int_a^b \Big[\Big(\frac{< u,Ru >^2 \, A < v,Rv >'}{< v,Rv >} \Big)' + A \, (< u,Ru >')^2$$

$$- A < v,Rv >' \Big(\frac{2 < u,Ru > < u,Ru >' < v,Rv > - < u,Ru >^2 < v,Rv >'}{< v,Rv >^2} \Big) \Big] dt$$

$$= \int_a^b \Big[A \, (< u,Ru >')^2 + \Big(\frac{< u,Ru >^2}{< v,Rv >} \; A < v,Rv >' \Big)'$$

$$- \Big(\frac{< u,Ru >^2}{< v,Rv >} \Big)' \, A < v,Rv >' \Big] dt$$

$$= \int_a^b \Big[A \, (< u,Ru >')^2 + \Big(\frac{< u,Ru >^2}{< v,Rv >} \Big) \, (A < v,Rv >')' \Big] dt$$

$$= \int_a^b \Big[A \, (< u,Ru >')^2 + \Big(\frac{< u,Ru >^2}{< v,Rv >} \Big) \, (A < v,Rv >')'$$

$$- C < u,Ru >^2 + \Big(\frac{< u,Ru >^2}{< v,Rv >} \Big) \, C < v,Rv > \Big] dt$$

$$= \int_a^b \Big[A \, (< u,Ru >')^2 - C < u,Ru >^2 + \Big(\frac{< u,Ru >^2}{< v,Rv >} \Big) \, L[v] \Big] dt$$

$$= \int_a^b \Big[A \, (< u,Ru >')^2 - C < u,Ru >^2 \Big] dt$$

$$= J[u] < 0$$

Thus, a contradiction arises from the assumption that $< v,Rv > \neq 0$. This means that $< v,Rv > = 0$ somewhere on $[a,b]$, which implies that $v(t) = 0$ for some $t \in [a,b]$.

Specific examples of $v(t)$ which satisfy $L[v]$ in Eq (4.7) are difficult to find. However, differential equations already in the form of Eq (4.7) may be examined.

<u>Example 4.1</u>. In the case where $v(t)$ is a scalar function of t, $R = 1$, $u(t) = t(h-t)$ on the interval $[0,h]$, and $A(t) = 1 - t^2$ and $C(t) = n(n+1)$ are coefficients from the Legendre polynomial examined in Chapter II, Eq (4.7) becomes

$$L[v] = [(1-t^2) \langle v(t), v(t) \rangle']' + n(n+1) \langle v(t), v(t) \rangle = 0 \qquad (4.10)$$

In order for $v(t)$ to have a zero on $[0,h]$, $J[u]$ defined in Eq (4.8) must be negative. Thus

$$J[u] = \int_0^h [(1-t^2)(\langle u,u \rangle')^2 - n(n+1) \langle u,u \rangle^2]dt$$

$$= \int_0^h [(1-t^2)(2 \langle u,u' \rangle)^2 - n(n+1) \langle u,u \rangle^2]dt$$

$$= \int_0^h [(1-t^2)\, 4(uu')^2 - n(n+1)\, (u^2)^2]dt$$

$$= \int_0^h [(1-t^2)\, 4(ht - t^2)^2(h - 2t)^2 - (n^2+ n)(ht - t^2)^4]dt$$

$$= \int_0^h [(4-4t^2)(h^4t^2- 6h^3t^3+ 13h^2t^4- 12ht^5+ 4t^6)$$

$$- (h^2+ n)(h^4t^4- 4h^3t^5+ 6h^2t^6- 4ht^7+ t^8)]dt$$

$$= \int_0^h [(4h^4t^2- 24h^3t^3+ 52h^2t^4- 48ht^5+ 16t^6)$$

$$- (4h^4t^4- 24h^3t^5+ 52h^2t^6- 48ht^7+ 16t^8)$$

$$- (n^2+ n)(h^4t^4- 4h^3t^5+ 6h^2t^6- 4ht^7+ t^8)]dt$$

$$= h^7\left(\frac{4}{3} - 6 + \frac{52}{5} - 8 + \frac{16}{7}\right) - h^9\left(\frac{4}{5} - 4 + \frac{52}{7} - 6 + \frac{16}{9}\right)$$

$$- h^9(n^2 + n)\left(\frac{1}{5} - \frac{2}{3} + \frac{6}{7} - \frac{1}{2} + \frac{1}{9}\right)$$

$$= h^7\left[\frac{2}{105} - h^2 \left(\frac{2}{315} + \frac{n^2 + n}{630} \right)\right] = \frac{h^7}{630}\left[12 - h^2 (n^2 + n + 4)\right] < 0$$

Therefore, a zero of $v(t)$ will occur on $[0,h]$ when $h > \left(\frac{12}{n^2 + n + 4}\right)^{\frac{1}{2}}$ and $h < 1$. When $n = 3$, a zero lies in $[0, .866]$; when $n = 4$, a zero lies in $[0, .707]$.

It may be difficult to find vector functions which satisfy $L[v] = 0$ in Eq (4.7). However, it is possible to substitute the differential inequality $L[v] \leq 0$ into Eq (4.7) and still derive Theorem 4.1 without altering the proof substantially. This allows a wider range of vector-valued functions $v(x)$ to satisfy the resulting inequality.

Other results by Jones have been established using different conditions on the vector-valued function $v(t)$ and the vector-valued trial functions $u(t)$ (Ref 16:86).

V. Conclusion

There are several computer applications of iterative techniques which will iterate toward zeros of a function, but the initial starting value for these schemes often must be chosen carefully. If the initial point is not close to an actual zero, the iterative techniques often diverge.

The theorems and examples presented in this work can be used to locate intervals on which a function has a zero. If the given function f can be paired with one of the differential equations $L[f] = 0$, and if a trial function u is found which makes the corresponding functional $J[u]$ negative on a certain interval, then a zero of f does exist on that interval.

Examples of common differential equations from different engineering sciences were chosen to demonstrate the breadth of applications possible. Perhaps this method can be applied to indicate the existence of zeros of other differential equations before valuable computer time is wasted in hit-or-miss searches to find these zeros. In fact, the algorithms obtained can themselves be implemented using numerical techniques on a computer of reasonable size.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This thesis extends the works of Morse, Leighton, and Jones which take a function defined by a second order differential equation and determine an interval on which that function either has a zero or attains a minimum, bounded value. Theorems for locating zeros are proved for functions of a single real variable, functions of two real variables, and real vector-valued functions. Algorithms suitable for computer adaptation are presented in examples which locate zeros of such functions occurring in the engineering sciences as Legendre polynomials, Laguerre polynomials, Emden's equation, and Duffing's equation; special emphasis

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is given to the zeros of Bessel functions $J_n(x)$. The methods developed in this work are useful in optimization theory; they also can be used to obtain good initial approximations of zeros for starting iterative algorithms, such as the Newton-Raphson method, which give more exact zero values.

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